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ABSTRACT

A general model is derived for the purpose of efficiently allocating integral numbers of units in multi-level designs given prespecified power levels. The derivation of the model is based on a constrained optimization problem that maximizes a general form of a ratio of expected mean squares subject to a budget constraint. This model provides more general closed form solutions than other available formulas. The proposed methodology allows for the determination of the optimal numbers of units for studies that involve more complex designs. In addition, the proposed model makes use of effect sizes and the estimates of variance components to optimize multi-level designs during the budget formulating stages. Thus, researchers are able to estimate the amount of currency needed to adjust power to a targeted level while precluding them from requesting too much or too little money in the budget planning phase of the study design. A procedure is also described for optimizing designs when estimates of variance components are unavailable. Case studies illustrate the methodology. Three appendixes contain supplemental calculations. (Contains 8 tables and 26 references.) (Author/SLD)

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# Optimizing Experimental Designs Relative to Costs and Effect Sizes

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**Optimizing Experimental Designs Relative to Costs and Effect Sizes**

A general model is derived for the purpose of efficiently allocating integral numbers of units in multi-level designs given prespecified power levels. The derivation of the model is based on a constrained optimization problem that maximizes a general form of a ratio of expected mean squares subject to a budget constraint. This model provides more general closed form solutions than other available formulae. As such, the proposed methodology allows for the determination of the optimal numbers of units for studies that involve more complex designs. Further, the proposed model makes use of effect sizes and the estimates of variance components to optimize multi-level designs during the budget formulating stages. Thus, researchers are able to estimate the amount of currency needed to adjust power to a targeted level while precluding them to request too much or too little money in the budget planning phase of the study design. A procedure is also described for optimizing designs when estimates of variance components are unavailable. Case studies are also provided to demonstrate and examine the methodology.

*Keywords*: Budget constraint; Effect size; Lagrange multiplier; Level of randomization; Multi-level design; Optimization; Power; Variance components

**Optimizing Experimental Designs Relative to Costs and Effect Sizes**

## 1. INTRODUCTION

The concern for the efficient allocation of economic resources in experimental designs has long been a topic of discussion (e.g., see Brooks, 1955; Cochran, 1977; Deming, 1953; Donner, Brown, & Brasher, 1990; Headrick & Zumbo, 2001; Hsieh, 1988; Marcoulides, 1993; Moerbeek, van Breukelen, & Berger. 2000, 2001; Muller, LaVange, Ramey, & Ramey, 1992; Overall & Dalal, 1965; Raudenbush, 1997; Snijders & Bosker, 1993). For example, Brooks (1955) derived a procedure for determining the optimal subsampling number subject to a budget constraint for a design that involved two-stage sampling. The constraint consisted of nonzero finite fixed prices for each input of the design (Brooks, 1955, Equation 2). This constraint was extended to three-stage sampling (Cochran, 1977, Equation 10.43) and also used to determine optimal sampling numbers for some multi-level experimental designs (e.g., Headrick & Zumbo, 2001, Equation 3; Moerbeek et al., 2000, Equation 6).

With this idea of an efficient allocation of resources in mind, let us first consider a (multi) $r$ level design where $c_i$ represents the nonzero fixed price for each of the $n_i$ units used at the $i$-th level of the design (where $i = 1,...,r$). For any particular $r$ level design, a number of different null hypotheses can be formulated and tested. However, in the test of any particular hypothesis, the form of the ratio of expected mean squares takes on a general form that can be expressed as follows:

$$\Phi = \frac{\sigma_o^2 + m\sigma_r^2}{\sigma_o^2}, \qquad (1a)$$

where $\Phi$ denotes the parametric form of the $F$ ratio. The term $\sigma_o^2$ is the total component

of variance present in both numerator and denominator. The term $\sigma_r^2$ is the variance component associated with some treatment effect. The term $m$ in the numerator is a general coefficient that denotes the total magnitude of all other coefficients associated with $\sigma_r^2$. The noncentrality parameter ($\delta$) associated with (1a) can be expressed as:

$$\delta = \frac{km\sigma_r^2}{\sigma_o^2}, \tag{2a}$$

where $k$ is the number of means (from $k$ populations) in the null hypothesis.

Consider, for example, a $r = 2$ level design with three factors: factor A consisting of $k$ populations each with $n_2$ second level units; factor B, crossed with factor A, and consisting of $n_1$ first level units; and factor C nested under factor B consisting of two types of observations for each of the $n_1$ units. Assuming factors B and C fixed and factor A random, the ratio of the expected mean squares in (1a) and the noncentrality parameter in (2a) for the omnibus test for a difference across the $k$ populations can be expressed as:

$$\Phi = \frac{\sigma_\varepsilon^2 + 2n_1\sigma_1^2 + 2n_1n_2\sigma_2^2}{\sigma_\varepsilon^2 + 2n_1\sigma_1^2}, \text{ and} \tag{1b}$$

$$\delta = \frac{k2n_1n_2\sigma_2^2}{\sigma_\varepsilon^2 + 2n_1\sigma_1^2}, \text{ where} \tag{2b}$$

$m = 2n_1n_2$, $\sigma_r^2 = \sigma_2^2$, and $\sigma_o^2 = \sigma_\varepsilon^2 + 2n_1\sigma_1^2$ in equations (1a) and (2a) above.

Because economic resources are scarce, a question to consider is what would be the optimal numbers of $n_1$ and $n_2$ to use in the two level design described above? More generally, this question can be formulated in terms of $r$ level designs and with respect to the additional concern for power as: *Given a finite budget, what are the optimal integral units of $n_1,...,n_i,...,n_r$ to use in a r level design such that the selected units of $n_i$ yield a*

*targeted level power at minimum cost?*

Overall and Dalal (1965) proposed a procedure for obtaining the optimal numbers of units ($n_i$) for some multi-level designs based on (1a) such that power could be maximized for a fixed budget. However, the problems with the Overall and Dalal (1965) procedure are its lack of generality and laborious nature. Specifically, the procedure suggested by Overall and Dalal (1965) requires the cumbersome task of writing out all possible experimental situations given fixed prices and a total budget. For each of the experimental possibilities, the subsequent tasks are (a) refer to previous research to estimate the variance components, (b) determine the noncentrality parameter, and (c) refer to power tables to determine the optimal scenario that maximizes power subject to the total budget. Further, Overall and Dalal (1965) made no attempt to target a particular level of power. As a result, the optimal solutions may yield an unacceptable level power because of an inadequate initial budget (e.g., the maximum power for a given initial budget could be .40).

The methodology of Brooks (1955), Cochran (1977), and Moerbeek et al. (2000) is based on minimizing error variance i.e., some specific forms of the denominator in (1a) to obtain the optimal sampling units of $n_i$. The problem with this method is that effect sizes, degrees of freedom, and power are not generally considered. Specifically, while these models do indeed minimize error variance, but similar to the procedure suggested by Overall and Dalal (1965), they may in general yield solutions with an unacceptable low level of power due to an insufficient budget. It should be noted that the procedure suggested by Moerbeek et al. (2000, p. 281) does allow for an effect size to be used to determine the minimum amount of dollars needed to achieve a certain level of power. However, their procedure is limited to two groups.

Moerbeek et al. (2000, Table 2) do offer convenient closed form solutions to obtain the optimal numbers of units in the context of two and three level designs. However, these equations are only applicable for some designs. For example, the equation for the solution to $n_1$ given in Moerbeek et al. (2000, Table 2, for two levels, and randomization at the class level) is not general enough to provide an optimum for $n_1$ above in equation (1b).

Because of the work of such researchers as Cohen (1988), Glass (1976), and Hedges (1981), some methodologists (e.g., Kirk, 1996; Rosenthal, Rosnow, & Rubin, 2000) are more concerned with the study and reporting of effect sizes - such as those related to $\sigma_\tau^2$ in (1a). Further, effect sizes are available (or can be estimated) to the investigator who is in the beginning stages of formulating a budget and selecting an appropriate experimental design (see, for example, Howell, 2000, p. 228).

## 2. PURPOSE OF THE STUDY

In view of the above, what is needed is a general model that can provide an estimate of the amount of adjustment to a budget necessary to bring power to a targeted level. More specifically, the purposes of the study are to (a) derive a procedure using Lagrange multipliers for determining the optimal numbers of units for multi-level designs with desired power targets, (b) provide more general closed form formulae that enable the determination of optimal numbers of units for studies that involve more complex designs, and (c) provide a method that optimizes multi-level designs when the estimates of variance components are unavailable. Case studies are provided to demonstrate the proposed procedure.

## 3. MATHEMATICAL DEVELOPMENT

### 3.1 When Estimates of Variance Components are Available

Let $\Phi$ in equation (1a) be defined in the context of a design with $r$ levels as follows:

$$\Phi = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2 + \cdots + p_r \left( \prod_{i=1}^{r} n_i \right) \sigma_r^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2 + \cdots + p_{r-1} \left( \prod_{i=1}^{r-1} n_i \right) \sigma_{r-1}^2}, \tag{3}$$

where $i = 1,\ldots,r$; the constants $p_i$ are nonnegative integers $\forall_{i<r}$; and the constant $p_r$ is a positive integer. Similar to the role of $m$ in (1a), the values of $p_i$ are general constant coefficients that denote the total magnitude of all other coefficients associated with their respective variance component ($\sigma_i^2$). The values of $n_1,\ldots,n_r$ in (3) are the variables of concern to select in such a manner that $\Phi$ is maximized subject to a budget constraint.

It can be shown that the ratio of expected mean squares in (1b) is a special case of (3) when randomization is performed at the (second) highest level. In general, equation (3) would appear with $r = 2$ as:

$$\Phi = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2}. \tag{4a}$$

Setting $p_1 = p_2 = 2$ gives equation (1b). If randomization were performed at the (first) lowest level then setting $p_1 = 0$ and $p_2 = 2$ in (4a) would give the appropriate expression for $\Phi$ as:

$$\Phi = \frac{\sigma_\varepsilon^2 + 2 n_1 n_2 \sigma_2^2}{\sigma_\varepsilon^2}. \tag{4b}$$

Let the total finite budget ($B$) associated with executing a study of the form in (3) be expressed as follows:

$$B = q_1 c_1 \prod_{j=1}^{r} n_j + q_2 c_2 \prod_{j=2}^{r} n_j + \cdots + q_i c_i \prod_{j=i}^{r} n_j + \cdots + q_r c_r n_r , \tag{5}$$

where $i = 1,...,r$, $j = i,...,r$, and the constants $q_i$ are positive integers. The values of $c_i$ represent the price per unit of $n_i$. The $q_i$ are general constant coefficients that are used to determine the total cost for each of the $r$ product terms (e.g., $c_i \prod_{j=i}^{r} n_j$) in (5).

Combining (3) and (5) more generally in terms of a Lagrangean expression is written as:

$$Z(n_1,...,n_r,\lambda) = f(n_1,...,n_r) + \lambda[B - g(n_1,...,n_r)], \tag{6}$$

where $\lambda$ is the Lagrange multiplier, $\Phi = f(n_1,...,n_r)$ is the objective function from (3) that is maximized subject to (5) for exogenous values of $c_i$, $p_i$, $q_i$, $\sigma_\varepsilon^2$, and $\sigma_i^2$ for all $i = 1,...,r$.

Without loss of generality, the optimal values of $n_i$ can be derived from setting $r = 2$ in (5) and subsequently substituting the right-hand sides of (4a) and (5) into the general framework of (6). The optimal solutions of $n_1$ ($n_1^*$), $n_2$ ($n_2^*$), and $\lambda$ ($\lambda^*$) are expressed as follows (see Appendix 1 for their derivations):

$$n_1^* = \frac{\sigma_\varepsilon}{\sigma_1}\sqrt{\frac{q_2 c_2}{p_1 q_1 c_1}}, \tag{7}$$

$$n_2^* = \frac{B\sigma_1\sqrt{p_1}}{\sigma_\varepsilon\sqrt{q_1 q_2 c_1 c_2} + \sigma_1 q_2 c_2 \sqrt{p_1}}, \text{ and} \tag{8}$$

$$\lambda^* = \frac{p_2\sigma_2^2}{(\sigma_\varepsilon\sqrt{q_1 c_1} + \sigma_1\sqrt{p_1 q_2 c_2})^2}. \tag{9}$$

In terms of microeconomic analysis, equations (7) and (8) imply that if the objective function in (6) is at a maximum (assuming that the objective function is quasi-

concave), then the marginal rate of substitution (*MRS*) of trading $n_1$ for $n_2$ must be equal to the price ratio $(c_1/c_2)$ of $n_1$ and $n_2$. This equality $MRS = c_1/c_2$ holds *iff* $n_1 = n_1^*$ and $n_2 = n_2^*$. It can be shown that $n_1^*$ and $n_2^*$ yield a maximum by showing that the objective function is quasi-concave on the domain consisting only of the nonnegative orthant (Arrow & Enthoven, 1961). See Appendix 2 for this derivation.

More generally, if randomization is performed at the (highest) $r$-th level for $r>2$, then in addition to $n_1^*$ from (7) the optimal solutions of $n_i^*$, $n_r^*$, and $\lambda^*$ are expressed as follows:

$$n_i^* = \frac{\sigma_{i-1}}{\sigma_i}\sqrt{\frac{p_{i-1}q_{i+1}c_{i+1}}{p_i q_i c_i}}\ , \ \forall_{2\le i\le r-1}, \tag{10}$$

$$n_r^* = \frac{B\sigma_{r-1}\sqrt{p_{r-1}}}{\sigma_\varepsilon\sqrt{q_1 q_r c_1 c_r} + \cdots + \sigma_i\sqrt{p_i q_{i+1} q_r c_{i+1} c_r} + \cdots + \sigma_{r-1}q_r c_r\sqrt{p_{r-1}}}, \text{ and} \tag{11}$$

$$\lambda^* = \frac{p_r\sigma_r^2}{(\sigma_\varepsilon\sqrt{q_1 c_1} + \sigma_1\sqrt{p_1 q_2 c_2} + \cdots + \sigma_{r-1}\sqrt{p_{r-1}q_r c_r})^2}. \tag{12}$$

If randomization is performed at the $i$-th lower level (for $i=2,\ldots,r-1$), then to solve for $n_i^*$ substitute equation (7), $i-2$ equations of the form in (10), and $r-i$ preassigned integers for all $n_i$ above the $i$-th level into the budget constraint of (6) and subsequently solve (6) for $n_i^*$. If randomization is performed at the lowest level ($i=1$), substitute $r-1$ preassigned integers for all $n_i$ above the first level into the budget constraint and then solve (6) for $n_1^*$. For convenience to the reader, presented in Table 1 through Table 3 are the formulae for the optimal solutions of $n_i^*$ for two, three, and four level designs at the various possible levels of randomization.

The solutions of $n_i^*$ given by equations (7), (10), and (11) require that they be integral numbers. However, these equations in general do not yield such numbers. Therefore, the following rule (Cameron, 1951) for rounding is used with respect to (7) and (10): if $n_i$ are positive integers such that $n_i < n_i^* < n_i + 1$ round up if $(n_i^*)^2 > n_i(n_i + 1)$ otherwise round down. The values of $n_i^*$ in (11) and Tables 1 through 3 are to be rounded in such a manner that the initial budget estimate ($B$) is not exceeded. Thus, the initial *realized* budget will be less than or equal to the initial estimate of $B$.

The Lagrange multiplier ($\lambda^*$) in (12) represents the *marginal* increase in the objective function ($\Phi$) of (6) given an exogenous increase of one dollar in the budget ($B$). Presented in Table 4 are the solutions for $\lambda^*$ in the context of two and three level designs under the various levels of randomization. Similar to Tables 1 through 3, the extension of the results in Table 4 to larger designs should be clear from the structure of the formulae.

### 3.2 Targeting a Level of Power

Given a design with randomization at the $i$-th level, the integral value of $n_i^*$ determined from (11) (or from Tables 1 through 3) may yield an undesirable level of power because of an inappropriate initial budget estimate. As such, we define $n_i^{(*)}$ as the integral number of units that yields the targeted level of power $\pi^{(*)}$ that is at least as large as a prespecified power threshold point denoted as $\pi^{(0)}$ (e.g., $\pi^{(0)} = .80$). More specifically, the targeted level of power $\pi^{(*)}$ is defined to be an element on the following interval:

$$\pi^{(-1)} < \pi^{(0)} \leq \pi^{(*)} < \pi^{(+1)}. \qquad (13)$$

The level of power denoted as $\pi^{(-1)}$ in (13) is associated with the $n_i^{(*)} - 1$ unit and considered undesirable because it falls below the prespecified minimum value of $\pi^{(0)}$. The upper limit of power denoted as $\pi^{(+1)}$ is associated with the $n_i^{(*)} + 1$ unit and considered to be an unnecessary amount of expenditures i.e., the marginal cost of an additional unit beyond $n_i^{(*)}$ exceeds the marginal benefit (or gain) in power.

To determine the amount of change to the initial realized budget such that power is at the level of $\pi^{(*)}$ in (13), the estimated $F$ ratio (denoted as $\hat{F}$ ) is first obtained. In general, $\hat{F}$ is computed by substituting the estimates of the variance components, the constant coefficients of $p_i$, and the optimal numbers of $n_1^*, \ldots, n_i^*, \ldots, n_r^*$ based on the initial realized budget into (3) as follows:

$$\hat{F} = \frac{\hat{\sigma}_\varepsilon^2 + p_1 n_1^* \hat{\sigma}_1^2 + p_2 n_1^* n_2^* \hat{\sigma}_2^2 + \cdots + p_r \left( \prod_{i=1}^{r} n_i^* \right) \hat{\sigma}_r^2}{\hat{\sigma}_\varepsilon^2 + p_1 n_1^* \hat{\sigma}_1^2 + p_2 n_1^* n_2^* \hat{\sigma}_2^2 + \cdots + p_{r-1} \left( \prod_{i=1}^{r-1} n_i^* \right) \hat{\sigma}_{r-1}^2}. \tag{14}$$

The critical point from the central $F$ distribution that corresponds with the degrees of freedom for $\hat{F}$ from (14) is subsequently used to determine $F^{(0)}$ which is the point on the noncentral $F$ distribution that yields power of $\pi^{(0)}$ in (13). The value of $F^{(0)}$ is computed by solving equation (30) in Appendix 3 for the noncentrality parameter ( $\delta^{(0)}$ ). It follows that:

$$F^{(0)} = 1 + \frac{\delta^{(0)}}{k}, \tag{15}$$

where $\delta^{(0)}$ is of the form in (2a). See Appendix 3 for further discussion and an example for computing $\delta^{(0)}$.

The change in the initial realized budget (denoted as $\Delta B$ ) is then determined as:

$$\Delta B = \frac{F^{(0)} - \hat{F}}{\lambda^*}. \qquad (16)$$

If $F^{(0)} > \hat{F}$ (or if $F^{(0)} < \hat{F}$) in (16), then $\Delta B$ is the minimum increase (or maximum decrease) to the total budget such that power remains greater than or equal to $\pi^{(0)}$ in (13).

Listed in Table 5 are the general relationships between the total budget expenditures and the various levels of power in (13). The value of $dB/dn_i$ in Table 5 is the derivative of the budget constraint in (5) with respect to $n_i$ (for randomization at the $i$-th level) and reflects the change in the total budget that is associated with a per unit change in $n_i$.

### 3.3 When the Estimates of Variance Components are Unavailable

In the absence of estimates of the variance components an approach the experimenter can take is to ask what would be a *minimum* effect size worth detecting. Using Cohen's (1988) definition of a standardized effect size ($f$) and a minimum estimated value of $\sigma_r$, the estimate of $\sigma_\varepsilon$ in (3) can be determined as follows:

$$\sigma_\varepsilon = \frac{\sigma_r}{f}, \text{ where } \sigma_r = \sqrt{\frac{\sum_k (\overline{Y} - \overline{\overline{Y}})^2}{k}}. \qquad (17)$$

A very useful contribution from Brooks (1955, Table 1) is the pilot study that demonstrated the wide range of values that $n_1$ in (7) may take and still maintain at least 90% precision of the true optimum ($n_1^*$). With respect to the Brooks (1955) study, Cochran (1977) noted, "Because of the flatness of the optimum, these [variance components] ratios need not be obtained with high accuracy...the wide interval between the lower and upper limits [than maintain 90% precision] is striking in nearly all cases" (p. 282). Similar points were also made by Moerbeek et al. (2000, p. 278) with respect to

the flatness of the optimal solutions.

In view of this point, it is convenient to estimate of variance components in the manner suggested by Cochran (1977) as follows:

$$\frac{\sigma_{i-1}}{\sigma_i} = \frac{\sqrt{1-\rho_i}}{\sqrt{\rho_i}}, \tag{18}$$

where $\rho_i$ is the intraclass correlation between the elements at the $i$-th level. Estimates of $\rho_i$ can be obtained from similar studies that report reliability estimates. For example, reliability coefficients such as KR-21 are often reported on instruments that take repeated or duplicate measures. Otherwise, if other intraclass correlations are unknown, the interval considered for estimating the variance components is $\rho_i \in [.05, .50]$. Values of $\rho_i$ outside this interval are considered as unusually low or high intraclass correlations (Cochran, 1977). Substituting estimates of $\rho_1$ and $\sigma_\varepsilon$ into (18) and solving for $\sigma_1$ in terms of a two level design (i.e., $i = 1$) gives:

$$\sigma_1 = \frac{\sigma_\varepsilon \sqrt{\rho_1}}{\sqrt{1-\rho_1}}, \tag{19}$$

For $r$ level designs, $\sigma_i$ can be determined from $\sigma_{i-1}$ as:

$$\sigma_i = \frac{\sigma_{i-1}\sqrt{\rho_i}}{\sqrt{1-\rho_i}}, \forall_{2 \le i \le r-1}. \tag{20}$$

## 4. CASE STUDIES

*Case 1.* Consider a researcher formulating a budget to study the effect of conditioned suppression on animal behavior. Of interest to the researcher is the $F$ test for differences between groups. The experimenter has data and research results reported by Howell (2000) of the effects of conditioned suppression on three groups of rats from a

three-factor design with repeated measures. (See Howell, 2000, p. 480-484 for the complete data set and a discussion on the effects of conditioned suppression. Variance component estimates were obtained from the data set using Minitab, 2000.)

The researcher desires to give each of the $n_2$ animals in the experiment $n_1$ repeated measures across four different cycles. Suppose the experimenter estimates an initial budget of $B = \$5250$ and desires a prespecified power level of at least $\pi^{(0)} = .80$. Presented in Table 5 is a summary of the steps and numerical information for determining the optimal solutions for this experiment. As indicated in Table 1, the optimal solutions are $n_1^* = 2$ repeated measures and $n_2^{(*)} = 14$ animals for each of $k = 3$ groups. Thus, the initial realized budget is required to be increased to $B = \$5880$ in order to achieve the targeted level of power ($\pi^{(*)}$).

*Case 2.* In a study investigating the source of error in viscosity measurements taken from shipping vessels, Hocking (1996) reported that the source of error was attributed to the differences between the vessels. The study was a two-factor fully nested ANOVA that included a single fluid.

A researcher is subsequently interested in investigating the effects of differences between fluids on the measurements of viscosity using $k = 2$ treatment groups. Based on the estimates of the variance components (Hocking, 1996, p. 611), the researcher has decided that a minimum value of $\sigma_3^2 = 0.125$ would be meaningful to detect with power prespecified to be at least $\pi^{(0)} = .80$. Assume that the researcher has \$150,000 available for this study. Presented in Table 7 are the estimates of the variance components as well as the other relevant information that summarizes this example. From Table 7, the optimal numbers for this example are $n_1^* = 2$ duplicate trials, $n_2^* = 2$ independent trials,

and $n_3^{(*)} = 4$ shipping vessels. A total budget of \$99,520 is required for the targeted power level with randomization at the shipping vessel level. Thus, only about two-thirds of the available money to the researcher is required to achieve the desired level of power.

*Case 3.* Suppose that a difference of 20 units between $k = 2$ groups (experimental and control) on a new treatment intervention and the standard treatment is considered meaningful to detect with power of at least $\pi^{(0)} = .90$. Using Cohen's (1988) definition of a medium effect size ( $f = 0.50$ ) and equation (17) gives:

$$\sigma_\varepsilon = \frac{\sigma_2}{f} = \frac{10}{.50} = 20.$$

Assume prior research indicates that the instrument taking the duplicate measures has good reliability, $\rho_1 = .80$. The value of $\rho_2$ is set equal to $\rho_2 = .25$ (the approximate midpoint of the intraclass correlation interval discussed in the previous section). Thus, from (19) and (20) the variance component estimates are:

$$\sigma_1 = \frac{20(\sqrt{.80})}{\sqrt{1-.80}} = 40, \text{ and}$$

$$\sigma_2 = \frac{40(\sqrt{.25})}{\sqrt{1-.25}} = 23.094.$$

An initial budget estimate is set to $B = \$12{,}500$. The various prices for each input of the design are listed in Table 8. As indicated in Table 8, the optimal numbers for this example are $n_1^* = 1$ (duplicate) measure, $n_2^* = 3$ independent trials, and $n_3^{(*)} = 65$ subjects. A total budget of \$36,400 is required with randomization at the subject level. As indicated in Table 8 the initial budget was substantially underestimated (\$12,500) for achieving the targeted level of power.

In practice, when the variance components are (*a priori*) unknown, it is prudent to

take an approach that yields conservative (biased downward) estimates of $\hat{F}$ in (16). The consequence of this approach is that it will require a larger total budget to ensure achieving a minimum targeted level of power.

## 5. CONCLUSION

The proposed procedure determines the efficient allocation of resources for experimental designs that include multi-stage sampling at a given level of randomization. The procedure is based on microeconomic analysis for determining optimal states. Specifically, given a fixed budget, various prices, and the estimates of variance components for a design, the Lagrange multiplier method locates the point where the ratio of expected mean squares is at a maximum. At the optimal state, each unit purchased will yield the same marginal utility per dollar spent on that unit. As a result, each unit will have an identical *marginal* cost to *marginal* benefit ratio.

The procedure presented simplifies the Overall and Dalal (1965) procedure for determining optimal solutions to the extent that there is no need to list all possible combinations of potential solutions. The example and discussion in Overall and Dalal (1965) on the special "zero-overhead" case for a simple repeated measures design is also subsumed under the proposed method. That is, if there is no cost associated with each subject (i.e., $c_2$ is arbitrarily close to zero in equations 7 and 8), then test each subject once and use as many subjects in the design that the budget allows.

From the case studies examined, it is evident that the procedure enables the researcher to estimate the amount of dollars needed to adjust power to some desired level. Using effect sizes and Lagrange multipliers from the model precludes the researcher from requesting either too much or too little money when formulating a budget.

**Table 1.** Solutions of $n_i^*$ for a two level design where randomization is at the $i$-th level.

| Level Randomized | Solution for $n_i^*$, $i=1, 2$ |
|---|---|
| 1 | $n_1^* = \dfrac{B - q_2 c_2 n_2}{q_1 c_1 n_2}$ |
| 2 | $n_2^* = \dfrac{B\sigma_1\sqrt{p_1}}{\sigma_\varepsilon\sqrt{q_1 q_2 c_1 c_2} + \sigma_1 q_2 c_2 \sqrt{p_1}}$ |

**Table 2.** Solutions of $n_i^*$ for a three level design where randomization is at the $i$-th level.

| Level Randomized | Solution for $n_i^*$, $i=1, 2, 3$ |
|---|---|
| 1 | $n_1^* = \dfrac{B - q_2 c_2 n_2 n_3 - q_3 c_3 n_3}{q_1 c_1 n_2 n_3}$ |
| 2 | $n_2^* = \dfrac{\sigma_1\sqrt{p_1}(B - q_3 c_3 n_3)}{\sigma_\varepsilon n_3\sqrt{q_1 q_2 c_1 c_2} + \sigma_1 q_2 c_2 n_3 \sqrt{p_1}}$ |
| 3 | $n_3^* = \dfrac{B\sigma_2\sqrt{p_2}}{\sigma_\varepsilon\sqrt{q_1 q_3 c_1 c_3} + \sigma_1\sqrt{p_1 q_2 q_3 c_2 c_3} + \sigma_2 q_3 c_3 \sqrt{p_2}}$ |

**Table 3.** Solutions of $n_i^*$ for a four level design where randomization is at the $i$-th level.

| Level Randomized | Solution for $n_i^*$, $i=1, 2, 3, 4$ |
|---|---|
| 1 | $n_1^* = \dfrac{B - q_2 c_2 n_2 n_3 n_4 - q_3 c_3 n_3 n_4 - q_4 c_4 n_4}{q_1 c_1 n_2 n_3 n_4}$ |
| 2 | $n_2^* = \dfrac{\sigma_1\sqrt{p_1}(B - q_3 c_3 n_3 n_4 - q_4 c_4 n_4)}{\sigma_\varepsilon n_3 n_4\sqrt{q_1 q_2 c_1 c_2} + \sigma_1 q_2 c_2 n_3 n_4 \sqrt{p_1}}$ |
| 3 | $n_3^* = \dfrac{\sigma_2\sqrt{p_2}(B - q_4 c_4 n_4)}{\sigma_\varepsilon n_4\sqrt{q_1 q_3 c_1 c_3} + \sigma_1 n_4\sqrt{p_1 q_2 q_3 c_2 c_3} + \sigma_2 q_3 c_3 n_4 \sqrt{p_2}}$ |
| 4 | $n_4^* = \dfrac{B\sigma_3\sqrt{p_3}}{\sigma_\varepsilon\sqrt{q_1 q_4 c_1 c_4} + \sigma_1\sqrt{p_1 q_2 q_4 c_2 c_4} + \sigma_2\sqrt{p_2 q_3 q_4 c_3 c_4} + \sigma_3 q_4 c_4 \sqrt{p_3}}$ |

**Table 4.** Solutions for the Lagrange multiplier ( $\lambda^*$ ) at various levels of randomization for two and three level designs.

| Level Randomized | Two level Design | Three level Design |
|---|---|---|
| 1 | $\lambda^* = \frac{p_2\sigma_2^2}{(\sigma_\varepsilon\sqrt{q_1c_1})^2}$ | $\lambda^* = \frac{p_3\sigma_3^2}{(\sigma_\varepsilon\sqrt{q_1c_1})^2}$ |
| 2 | $\lambda^* = \frac{p_2\sigma_2^2}{(\sigma_\varepsilon\sqrt{q_1c_1}+\sigma_1\sqrt{p_1q_2c_2})^2}$ | $\lambda^* = \frac{p_3\sigma_3^2}{(\sigma_\varepsilon\sqrt{q_1c_1}+\sigma_1\sqrt{p_1q_2c_2})^2}$ |
| 3 | ----- | $\lambda^* = \frac{p_3\sigma_3^2}{(\sigma_\varepsilon\sqrt{q_1c_1}+\sigma_1\sqrt{p_1q_2c_2}+\sigma_2\sqrt{p_2q_3c_3})^2}$ |

**Table 5.** The relationship between power and total budget expenditures. The power levels are associated with equation (13). $B$ represents the initial realized budget. The $\Delta B$ denotes the necessary change to $B$ to bring power to a minimum acceptable level of $\pi^{(0)}$. The number $n_i^*$ is based on $B$. The value of $n_i^{(*)}$ is the optimal number to bring power to the targeted level of $\pi^{(*)}$. Randomization is at the $i$-th level.[1]

| Power Level | Power Level and Expenditures | Total Budget |
|---|---|---|
| $\pi^{(-1)}$ | Insufficient | $B + [(n_i^{(*)} - 1) - n_i^*]\frac{dB}{dn_i}$ |
| $\pi^{(0)}$ | Minimum | $B + \Delta B$ |
| $\pi^{(*)}$ | Optimal | $B + [n_i^{(*)} - n_i^*]\frac{dB}{dn_i}$ |
| $\pi^{(+1)}$ | Excessive | $B + [(n_i^{(*)} + 1) - n_i^*]\frac{dB}{dn_i}$ |

[1]*Note*: Minimum and optimal levels are equal (i.e., $\pi^{(0)} = \pi^{(*)}$) *iff* $\Delta B = [n_i^{(*)} - n_i^*]\frac{dB}{dn_i}$.

**Table 6.** Summary of *Case 1*. The optimal numbers are $n_1^* = 2$ repeated measures, $n_2^{(*)} = 14$ animals for each of the $k = 3$ (treatment) groups, and with a total expenditures of \$5,880. Randomization is at the animal (second) level.

1. Lagrangean expression:

$$Z = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2} + \lambda[B - q_1 n_1 n_2 c_1 - q_2 n_2 c_2]$$

   1(a). Cost for each animal to enter the experiment: $c_2 = \$100$

   1(b). Cost for each repeated measure: $c_1 = \$5$

   1(c). Initial budget estimate: $B = \$5{,}250$

   1(d). Variance component estimates: $\sigma_\varepsilon^2 = 0.01908$, $\sigma_1^2 = 0.00697$, $\sigma_2^2 = 0.00244$

   1(e). Integer values of $p_i$ and $q_i$: $p_1 = 4$, $p_2 = 4$, $q_1 = 12$, $q_2 = 3$

2. Minimum power threshold point: $\pi^{(0)} = .80$
3. Initial realized budget: $B = \$5040 = (12)(2)(12)(\$5) + (3)(12)(\$100)$
4. Optimal integer solutions based on the initial realized budget:

   3(a). Number of repeated measures (eq. 7): $n_1^* = 2$

   3(b). Number of animals per group (Table 1, Level 2): $n_2^* = 12$

5. $\frac{dB}{dn_2} = \$420 = (12)(2)(\$5) + (3)(\$100)$
6. Lagrange multiplier (Table 4): $\lambda^* = 0.000622$
7. Values of $\hat{F}$ and $F^{(0)}$:

   7(a). For $p_1 = 4$, $p_2 = 4$, $n_1^* = 2$, $n_2^* = 12$ (eq. 14): $\hat{F} = 4.13$

   7(b). For $\pi^{(0)} = .80$ (eq. 15 and App. 3 with $z^{(0)} = -0.84$): $F^{(0)} = 4.47$
8. Change to the initial realized budget (eq. 16): $\Delta B = \$547$
9. Total expenditures associated with the power points $\pi^{(-1)}$, $\pi^{(0)}$, $\pi^{(*)}$, $\pi^{(+1)}$ in eq. (13) where $n_3^{(*)} = 14$ yields the targeted level of power $\pi^{(*)}$ (see Table 5):

| | | |
|---|---|---|
| 9(a). $\pi^{(-1)}$: | Insufficient | $\$5040 + [13 - 12] \times \$420 = \$5{,}460$ |
| 9(b). $\pi^{(0)} = .80$: | Minimum | $\$5040 + \$547 = \$5{,}587$ |
| 9(c). $\pi^{(*)}$: | **Optimal** | **$\$5040 + [14 - 12] \times \$420 = \$5{,}880$** |
| 9(d). $\pi^{(+1)}$: | Excessive | $\$5040 + [15 - 12] \times \$420 = \$6{,}300$ |

**Table 7.** Summary of *Case 2*. The optimal numbers are $n_1^* = 2$ duplicate trials, $n_2^* = 2$, trials, $n_3^{(*)} = 4$ shipping vessels for each of the $k = 2$ (experimental and control) groups, and with a total expenditures of \$99,520. Randomization is at the vessel (third) level.

1. Lagrangean expression:

$$Z = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2 + p_3 n_1 n_2 n_3 \sigma_3^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2} + \lambda[B - q_1 n_1 n_2 n_3 c_1 - q_2 n_2 n_3 c_2 - q_3 n_3 c_3]$$

   1(a). Cost for each shipping vessel: $c_3 = \$9{,}900$

   1(b). Cost for each trial: $c_2 = \$770$

   1(c). Cost for duplicating each trial: $c_1 = \$250$

   1(d). Initial budget estimate: $B = \$150{,}000$

   1(e). Variance component estimates: $\sigma_\varepsilon^2 = 0.03470$, $\sigma_1^2 = 0.02670$, $\sigma_2^2 = 0.08180$, $\sigma_3^2 = 0.125$

   1(f). Integer values of $p_i$ and $q_i$: $p_1 = 1$, $p_2 = 1$, $p_3 = 1$, $q_1 = 2$, $q_2 = 2$, $q_3 = 2$

2. Minimum power threshold point: $\pi^{(0)} = .80$

3. Initial realized budget:

   $B = \$149{,}280 = (2)(2)(2)(6)(\$250) + (2)(2)(6)(\$770) + (2)(6)(\$9900)$

4. Optimal integer solutions based on the initial realized budget:

   3(a). Number of duplicated trials (eq. 7): $n_1^* = 2$

   3(b). Number of trials per vessel (eq. 10): $n_2^* = 2$

   3(c). Number of vessels per group (Table 2, Level 3): $n_3^* = 6$

5. $\dfrac{dB}{dn_3} = \$24{,}880 = (2)(2)(2)(\$250) + (2)(2)(\$770) + (2)(\$9900)$

6. Lagrange multiplier (Table 4): $\lambda^* = 0.0000484$

7. Values of $\hat{F}$ and $F^{(0)}$:

   7(a). For $p_1 = p_2 = p_3 = 1$, $n_1^* = 2$, $n_2^* = 2$, $n_3^* = 6$ (eq. 14): $\hat{F} = 8.22$

   7(b). For $\pi^{(0)} = .80$ (eq. 15 and App. 3 with $z^{(0)} = -0.84$): $F^{(0)} = 5.75$

8. Change to the initial realized budget (eq. 16): $\Delta B = -\$51{,}033$

9. Total expenditures associated with the power points $\pi^{(-1)}$, $\pi^{(0)}$, $\pi^{(*)}$, $\pi^{(+1)}$ in eq. (13) where $n_3^{(*)} = 4$ yields the targeted level of power $\pi^{(*)}$ (see Table 5):

| | | |
|---|---|---|
| 9(a). $\pi^{(-1)}$: | Insufficient | $\$149{,}280 + [3-6] \times \$24{,}880 = \$74{,}640$ |
| 9(b). $\pi^{(0)} = .80$: | Minimum | $\$149{,}280 + (-\$51{,}033) = \$98{,}247$ |
| 9(c). $\pi^{(*)}$: | **Optimal** | **$\$149{,}280 + [4-6] \times \$24{,}880 = \$99{,}520$** |
| 9(d). $\pi^{(+1)}$: | Excessive | $\$149{,}280 + [5-6] \times \$24{,}880 = \$124{,}400$ |

**Table 8.** Summary of *Case 3*. The optimal numbers are $n_1^* = 1$ duplicate measures, $n_2^* = 3$ repeated measures, $n_3^{(*)} = 65$ subjects for each of $k = 2$ groups, and with a total expenditures of \$36,400. Randomization is at the subjects (third) level.

1. Lagrangean expression:

$$Z = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2 + p_3 n_1 n_2 n_3 \sigma_3^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2} + \lambda [B - q_1 n_1 n_2 n_3 c_1 - q_2 n_2 n_3 c_2 - q_3 n_3 c_3]$$

   1(a). Cost for training each subject: $c_3 = \$100$

   1(b). Cost for each test occasion: $c_2 = \$50$

   1(c). Cost for duplicating each test: $c_1 = \$10$

   1(d). Initial budget estimate: $B = \$12{,}500$

   1(e). Variance component estimates: $\sigma_\varepsilon^2 = 20,\ \sigma_1^2 = 40,\ \sigma_2^2 = 23.094,\ \sigma_3^2 = 100$

   1(f). Integer values of $p_i$ and $q_i$: $p_1 = 1\ \ p_2 = 1\ \ p_3 = 1\ \ q_1 = 2\ \ q_2 = 2\ \ q_3 = 2$

2. Minimum power threshold point: $\pi^{(0)} = .90$

3. Initial realized budget:

   $B = \$12{,}320 = (2)(1)(3)(22)(\$10) + (2)(3)(22)(\$50) + (2)(22)(\$100)$

4. Optimal integer solutions based on the initial realized budget:

   4(a). Number of duplicated trials (eq. 7): $n_1^* = 1$

   4(b). Number of test occasions (eq. 10): $n_2^* = 3$

   4(c). Number of subjects per group (Table 2, Level 3): $n_3^* = 22$

5. $\dfrac{dB}{dn_3} = \$560 = (2)(1)(3)(\$10) + (2)(3)(\$50) + (2)(\$100)$

6. Lagrange multiplier (Table 4): $\lambda^* = 0.000150$

7. Values of $\hat{F}$ and $F^{(0)}$:

   7(a). For $p_1 = p_2 = p_3 = 1$, $n_1^* = 1$, $n_2^* = 3$, $n_3^* = 22$ (eq. 14): $\hat{F} = 2.83$

   7(b). For $\pi^{(0)} = .90$ (eq. 15 and App. 3 with $z^{(0)} = -1.28$ ): $F^{(0)} = 6.39$

8. Change to the initial realized budget (eq. 16): $\Delta B = \$23{,}733$

9. Dollar values associated with the power points $\pi^{(-1)}$, $\pi^{(0)}$, $\pi^{(*)}$, $\pi^{(+1)}$ in eq. (13) where $n_3^{(*)} = 65$ yields the targeted level of power $\pi^{(*)}$ (see Table 5):

| | | |
|---|---|---|
| 9(a). $\pi^{(-1)}$: | Insufficient | $\$12{,}320 + [64 - 22] \times \$560 = \$35{,}840$ |
| 9(b). $\pi^{(0)} = .90$: | Minimum | $\$12{,}320 + (\$23{,}733) = \$36{,}053$ |
| 9(c). $\pi^{(*)}$: | **Optimal** | **$\$12{,}320 + [65 - 22] \times \$560 = \$36{,}400$** |
| 9(d). $\pi^{(+1)}$: | Excessive | $\$12{,}320 + [66 - 22] \times \$560 = \$36{,}960$ |

## APPENDIX 1.

The specific form of the Lagrangean expression in (6) for $r=2$ is expressed as:

$$Z = \frac{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2 + p_2 n_1 n_2 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 c_1 \sigma_1^2} + \lambda[B - q_1 c_1 n_2 n_1 - q_2 c_2 n_2]. \tag{21}$$

Differentiating (21) with respect to $n_1$, $n_2$, and $\lambda$ and equating the partial derivatives to zero yields:

$$Z_{n_1} = \frac{\partial Z}{\partial n_1} = 0 = -\frac{n_2(c\lambda_1(q_1\sigma_\varepsilon^4 + p_1^2 q_1 n_1^2 \sigma_1^4 + 2p_1 q_1 n_1 \sigma_1^2 \sigma_\varepsilon^2) - p_2 \sigma_\varepsilon^2 \sigma_2^2)}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}, \tag{22a}$$

$$Z_{n_2} = \frac{\partial Z}{\partial n_2} = 0 = -\lambda(q_1 n_1 c_1 + b_2 c_2) + \frac{p_2 n_1 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2}, \text{ and} \tag{22b}$$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0 = B - q_1 c_1 n_2 n_1 - q_2 c_2 n_2. \tag{22c}$$

Equation (22c) returns the budget constraint. Solving (22a) and (22b) for $\lambda$ gives:

$$\lambda = \frac{p_2 \sigma_\varepsilon^2 \sigma_2^2}{q_1 c_1 (\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}, \text{ and} \tag{23a}$$

$$\lambda = \frac{p_2 n_1 \sigma_2^2}{(q_1 n_1 c_1 + q_2 c_2)(p_1 n_1 \sigma_1^2 + \sigma_\varepsilon^2)}. \tag{23b}$$

Dividing (23a) by (23b) and multiplying through both sides by the price ratio ($c_1/c_2$) yields:

$$\frac{c_1}{c_2} = \frac{\sigma_\varepsilon^2 (q_1 n_1 c_1 + q_2 c_2)}{q_1 n_1 c_2 (\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)}. \tag{24}$$

Solving (24) for $n_1$ gives the optimal number of $n_1^*$ in equation (7). Subsequently substituting the right-hand side of (7) into the constraint of (21) and into (23a) gives the optimal numbers of $n_2^*$ and $\lambda^*$ in equations (8) and (9) respectively.

## APPENDIX 2.

The first and second-order partial derivatives of $\Phi$ (the objective function) in (21) with respect to $n_1$ and $n_2$ are as follows:

$$\frac{\partial \Phi}{\partial n_1} = f_{n_1} = \frac{p_2 n_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}, \tag{25}$$

$$\frac{\partial \Phi}{\partial n_2} = f_{n_2} = \frac{p_2 n_1 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2}, \tag{26}$$

$$\frac{\partial^2 \Phi}{\partial n_1^2} = f_{n_1 n_1} = -\frac{2 p_1 p_2 n_2 \sigma_\varepsilon^2 \sigma_1^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^3}, \tag{27}$$

$$\frac{\partial^2 \Phi}{\partial n_2^2} = f_{n_2 n_2} = 0, \text{ and} \tag{28}$$

$$\frac{\partial^2 \Phi}{\partial n_1 n_2} = \frac{\partial^2 \Phi}{\partial n_2 n_1} = f_{n_1 n_2} = f_{n_2 n_1} = \frac{p_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}. \tag{29}$$

The determinantal test for quasi-concavity is (Chiang, 1984, p. 394):

$$|\overline{\Phi}_1| = \begin{vmatrix} 0 & f_{n_1} \\ f_{n_1} & f_{n_1 n_1} \end{vmatrix} = -\left(\frac{\partial \Phi}{\partial n_1}\right)^2 < 0, \text{ and}$$

$$|\overline{\Phi}_2| = \begin{vmatrix} 0 & f_{n_1} & f_{n_2} \\ f_{n_1} & f_{n_1 n_1} & f_{n_1 n_2} \\ f_{n_2} & f_{n_2 n_1} & f_{n_2 n_2} \end{vmatrix} = 2\left(\frac{\partial \Phi}{\partial n_1} \cdot \frac{\partial \Phi}{\partial n_2} \cdot \frac{\partial^2 L}{\partial n_2 n_1}\right) - \left(\frac{\partial \Phi}{\partial n_2}\right)^2 \left(\frac{\partial^2 \Phi}{\partial n_1^2}\right) - \left(\frac{\partial \Phi}{\partial n_1}\right)^2 \left(\frac{\partial^2 \Phi}{\partial n_2^2}\right) > 0.$$

On substitution of (25) through (29), it follows that:

$$|\overline{\Phi}_1| = -\left(\frac{p_2 n_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}\right)^2 < 0, \text{ and}$$

$$|\overline{\Phi}_2| = 2\left(\frac{p_2 n_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2} \cdot \frac{p_2 n_1 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2} \cdot \frac{p_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}\right) -$$

$$\left(\frac{p_2 n_1 \sigma_2^2}{\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2}\right)^2 \left(-\frac{2 p_1 p_2 n_2 \sigma_\varepsilon^2 \sigma_1^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^3}\right) - \left(\frac{p_2 n_2 \sigma_\varepsilon^2 \sigma_2^2}{(\sigma_\varepsilon^2 + p_1 n_1 \sigma_1^2)^2}\right)^2 (0) > 0,$$

because $n_1$, $n_2$, and $p_2$ are positive and $p_1$ is nonnegative. Hence, $\Phi$ is quasi-concave.

## APPENDIX 3.

The value of $F^{(0)}$ in equation (15) is based on the following expression (Winer, 1991, p. 136):

$$z^{(0)} = \frac{\sqrt{(2\upsilon_2 - 1)\frac{\upsilon_1 F_0}{\upsilon_2}} - \sqrt{(2\upsilon_1 - \delta)\frac{\upsilon_1 + 2\delta}{\upsilon_1 + \delta}}}{\sqrt{\frac{\upsilon_1 F_0}{\upsilon_2} + \frac{\upsilon_1 + 2\delta}{\upsilon_1 + \delta}}}, \tag{30}$$

where $z^{(0)}$ is unit normal, $\delta$ is a noncentrality parameter, and $\upsilon_1$ and $\upsilon_2$ are the degrees of freedom associated with the numerator and denominator of the $F$ ratio. If $F_0$ in (30) is a point on the $F(\upsilon_1, \upsilon_2, \delta)$ distribution, then the $\Pr(F > F_0) \doteq \Pr(z > z^{(0)})$. Setting $z^{(0)}$ in (30) to the appropriate value associated with minimum power of $\pi^{(0)}$ in (13) and solving for $\delta$ gives $\delta^{(0)}$. Because $\delta^{(0)}$ is of the form in (2a) it follows that:

$F^{(0)} = 1 + \frac{\delta^{(0)}}{k}$ which appears in equation (15).

As an example, from the initial realized budget in case 1, we have degrees of freedom of $\upsilon_1 = 2$, $\upsilon_2 = 33$, and a critical point of $F_0 = 3.28$. Because $\pi^{(0)} = .80$ in (13) we set $z^{(0)} = -0.84$. Substituting these values into (30) gives:

$$-0.84 = \frac{\sqrt{\frac{[2(33)-1](2)(3.28)}{33}} - \sqrt{[2(2)-\delta]\frac{2+2\delta}{2+\delta}}}{\sqrt{\frac{(2)(3.28)}{33} + \frac{2+2\delta}{2+\delta}}}. \tag{31}$$

Numerically solving (31) for $\delta$, using the equation solver FindRoot (Wolfram, *Mathematica*, version 4.0, 1999), gives $\delta^{(0)} = 10.41$. Thus, given that $k = 3$ for this case we have $F^{(0)} = 1 + \frac{\delta^{(0)}}{k} = 1 + \frac{10.41}{3} = 4.47$.

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